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# Improved Fixed Interface Method for Modal Synthesis

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**The paper takes a fresh look at the important topic of substructure synthesis. An improved technique for synthesizing the modes obtained with fixed interface coordinates is proposed. The reduction in the size of a substructure model is achieved through a higher order mode superposition approach. For the first time, an analytical rationale for the popular Craig-Bampton-Hurty method is presented. It is shown that the Craig-Bampton-Hurty method is a special case of the proposed method. The proposed approach can be used to improve the accuracy of the calculated eigenproperties by utilizing the higher order component modes but without calculating additional normal modes of the substructures. Numerical results are presented to demonstrate the effectiveness of the proposed method with the increasing order of the mode superposition approach used. The approach is expected to be efficient if the degrees of freedom at the interface of the substructures are relatively small compared with the degrees of freedom of the substructures.**

## Introduction

FOR forced response calculations, the lower modes of vibration of a structure are usually quite adequate to characterize the behavior of the system with reasonable accuracy. Thus, the use of model reduction techniques to calculate the lower modes has become a standard practice in structural dynamics. The two techniques that are commonly used to reduce the size of a dynamic model are 1) dynamic condensation or dynamic reduction methods and 2) modal synthesis or component mode methods. This paper is concerned with the development of a new modal synthesis method.

A modal synthesis method is an analytical process to synthesize the lower modal properties of a structure from those of its individual substructures through a Ritz procedure. The technique requires four basic steps:

1) The structural system is first partitioned into smaller substructures. Such divisions, if they are not dictated by contractual agreements, are usually done along the boundaries of components with different functions, geometric, or dynamic characteristics.

2) The physical displacements of the substructures are transformed into a reduced set of generalized coordinates through a matrix of component modes. The component modes are a combination of the lower normal modes of vibration and a set of Ritz vectors that are introduced to minimize the errors due to the use of a reduced number of vibration modes.[1]

3) The substructures' equations of motion in terms of the reduced generalized coordinates are coupled to obtain the global equations of motion of the combined system. This is done by enforcing the compatibility and equilibrium conditions along the component interfaces.

4) The solution of the eigenvalue problem of reduced order is obtained to get the lower approximate eigenproperties of the complete structural system.

There are several methods for modal synthesis in the present literature. The differences between these methods are primarily due to the variations in the selection of different types of modes as the component modes. To obtain the normal vibration modes, one can select different kinematic conditions for the degrees of freedom at which a substructure is connected to the remainder of the structure. According to the interface conditions used, the modal synthesis formulations can be classified[2] as fixed interface methods,[3-5] free interface methods,[6-9] hybrid methods,[7] or loaded interface methods.[10]

The first modal synthesis technique introduced by Hurty[3] was a fixed interface method. Three types of component modes were used in Hurty's original method: rigid-body modes, constrained or attachment modes, and fixed interface normal vibration modes. The method was later modified by Craig and Bampton.[4] Only the constrained and normal vibration modes were used in their formulation since they showed that there is no need to separately use the rigid-body modes. This modified version, which will be referred to as the Craig-Bampton-Hurty (CBH) method in this paper, is one of the most well-known and frequently used modal synthesis techniques. Its straightforward formulation and the simplicity of the stiffness and mass matrices in the synthesized eigenproblem contribute to its popularity in the profession. Another advantage of the method is that the cases of restrained or unrestrained substructures can be treated with the same ease since the substructures with fixed interface do not posses rigid-body modes that often may introduce additional complications. On the other hand, comparison studies[11,12] have shown the CBH method is not able to provide the combined system's eigenproperties with the same accuracy that some of the free interface methods, such as the Craig and Chang method,[11] can.

It would be useful if one could improve the accuracy of the results obtained with the CBH technique while at the same time retaining the convenient features of a fixed interface approach. In this paper, therefore, a new fixed interface method is presented. The proposed method is based on the concepts of the so-called higher order modal combination techniques.[13-16] The higher order modal combination techniques are not new, and their origin can be traced to Bisplinghoff.[17] However, what is new here is the utilization of this technique for the development of an efficient modal synthesis approach. These higher order modal combination methods basically provide a systematic way to accelerate the convergence of the modal summation process while at the same time retaining the minimum number of modes. The mode acceleration method[12] is the most well-known example of the higher order modal combination approaches.

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Here, it is shown that the CBH method is a particular case of the proposed method. With the help of the formulation presented herein, the implicit approximations inherent in the CBH method can now be analytically explained. Numerical examples show that the improved method is able to predict accurate values of the natural frequencies and vibration modes with the use of only limited substructure modes.

## Substructure Motion in Terms of Fixed Interface Modes

Consider the structural system shown in Fig. 1 divided into two substructures. We seek to determine its modal properties by synthesizing the corresponding properties of its substructures, calculated with the interface coordinates fixed. To develop a modal synthesis technique to carry out this task, a formulation that will allow the motion of the substructures to be expressed in terms of a reduced number of generalized coordinates is presented first.

Let the physical coordinates $u$ of a substructure be divided into a set of "interior" coordinates $u_I$ and a set of interface or "juncture" coordinates $u_J$. The number of interior degrees of freedom (DOF) is $n_I$, and the number of interface DOF is $n_J$. The equations of motion of this substructure in matrix form can be written as

$$\begin{bmatrix} M_{II} & M_{IJ} \\ M_{JI} & M_{JJ} \end{bmatrix} \begin{Bmatrix} \ddot{u}_I \\ \ddot{u}_J \end{Bmatrix} + \begin{bmatrix} K_{II} & K_{IJ} \\ K_{JI} & K_{JJ} \end{bmatrix} \begin{Bmatrix} u_I \\ u_J \end{Bmatrix} = \begin{Bmatrix} f_I \\ f_J \end{Bmatrix} \tag{1}$$

Since we are interested in calculating the free vibration characteristics, the forced vector $f_I$ associated with the internal coordinates can be set equal to zero. From the first $n_I$ rows of Eq. (1) one can then write

$$[M_{II}]\ddot{u}_I + [K_{II}]u_I = -[M_{IJ}]\ddot{u}_J - [K_{IJ}]u_J \tag{2}$$

Using the mode superposition approach, we can solve Eq. (2) for $u_I$ as

$$u_I = \sum_{j=1}^{n_I} \phi_j \eta_j(t) \tag{3}$$

where $\phi_j$ is the $j$th eigenvector obtained as the solution of the following eigenvalue problem associated with the homogeneous part of Eq. (2),

$$[K_{II} - \lambda_j M_{II}]\phi_j = 0, \qquad j = 1, \ldots, n_I \tag{4}$$

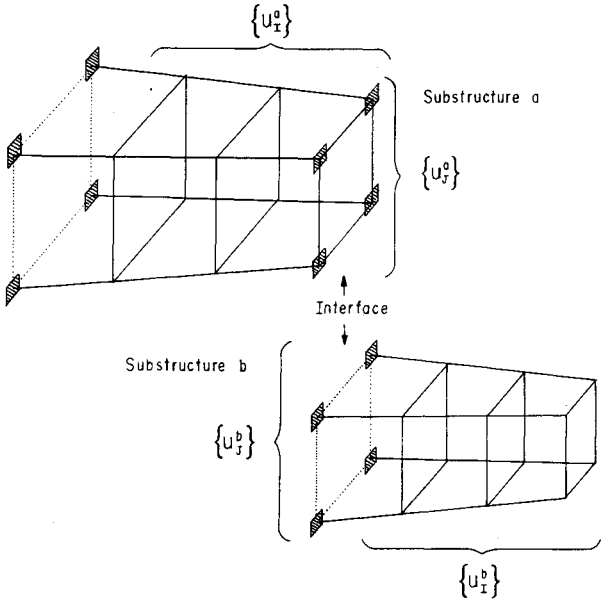


**Fig. 1 Substructures with fixed interface coordinates.**

and $\eta_j(t)$ is the principal coordinate that is obtained from the decoupled equation,

$$\ddot{\eta}_j(t) + \lambda_j \eta_j(t) = P_j(t), \qquad j = 1, \ldots, n_I \tag{5}$$

The generalized force $P_j(t)$ is defined in terms of the right-hand side of Eq. (2) as

$$P_j(t) = -\phi_j^T([M_{IJ}]\ddot{u}_J + [K_{IJ}]u_J) \tag{6}$$

Here it is assumed that the modes $\phi_j$ are mass normalized as

$$\phi_j^T[M_{II}]\phi_k = \delta_{jk} \tag{7}$$

It is relevant to note that the modes defined by eigenvalue problem (4) are the normal modes of the substructure with its interface degrees of freedom considered fixed.

For quiescent initial conditions, the solution of Eq. (5) is

$$\eta_j(t) = \frac{1}{\omega_j} \int_0^t P_j(\tau) \sin \omega_j(t - \tau)\, d\tau \tag{8}$$

Separating the terms in the summation in Eq. (3) into two sets involving the first lowest $n_k$ modes and the higher $n_R = n_I - n_K$ modes and substituting Eq. (8) in the second summation, we can write for $u_I$ as follows:

$$u_I = \sum_{j=1}^{n_K} \phi_j \eta_j(t) + \sum_{j=n_K+1}^{n_I} \phi_j \frac{1}{\omega_j} \int_0^t P_j(\tau) \sin \omega_j(t - \tau)\, d\tau \tag{9}$$

The convolution integral in Eq. (9) can be integrated by parts to yield

$$\int_0^t P_j(\tau) \sin \omega_j(t - \tau)\, d\tau = \frac{P_j(t)}{\omega_j} - \frac{P_j(0)}{\omega_j} \cos \omega_j t - \frac{1}{\omega_j} \int_0^t \dot{P}_j(\tau) \cos \omega_j(t - \tau)\, d\tau \tag{10}$$

After two more integrations by parts, the previous expression can be written as

$$\int_0^t P_j(\tau) \sin \omega_j(t - \tau)\, d\tau = \frac{P_j(t)}{\omega_j} - \frac{\ddot{P}_j(t)}{\omega_j^3} - \frac{1}{\omega_j} \epsilon(t) + \frac{1}{\omega_j^3} \nu(t) \tag{11}$$

with

$$\epsilon(t) = P_j(0) \cos \omega_j t + \frac{\dot{P}_j(0)}{\omega_j} \sin \omega_j t - \frac{\ddot{P}_j(0)}{\omega_j^2} \cos \omega_j t \tag{12}$$

$$\nu(t) = \int_0^t D^2 \dot{P}_j(\tau) \cos \omega_j(t - \tau)\, d\tau \tag{13}$$

where we have introduced the notation of $D^2 \equiv (d^2/dt^2)$ to avoid the use of several overdots to represent time differentiation. Replacing back Eqs. (12) and (13) in Eq. (9), the vector of internal displacements can be written as

$$u_I = \sum_{j=1}^{n_K} \phi_j \eta_j(t) + \sum_{j=n_K+1}^{n_I} \phi_j \left[ \frac{P_j(t)}{\omega_j^2} - \frac{\ddot{P}_j(t)}{\omega_j^4} \right] - \sum_{j=n_K+1}^{n_I} \phi_j \left[ \frac{\epsilon(t)}{\omega_j^2} - \frac{\nu(t)}{\omega_j^4} \right] \tag{14}$$

When the dynamic response is calculated by modal analysis, it is customary to retain only the lower $n_K$ modes in the expansion. A more accurate approximation can be obtained by also retaining the second term in Eq. (14). That is,

$$u_I \simeq \sum_{j=1}^{n_K} \phi_j \eta_j(t) + \sum_{j=n_K+1}^{n_I} \phi_j \left[ \frac{P_j(t)}{\omega_j^2} - \frac{\ddot{P}_j(t)}{\omega_j^4} \right] \tag{15}$$

Recalling Eq. (6), we can rewrite Eq. (15) as

$$u_I \simeq \sum_{j=1}^{n_K} \phi_j \eta_j(t) - \sum_{j=n_K+1}^{n_I} \frac{\phi_j \phi_j^T}{\omega_j^2} \{[M_{IJ}]\ddot{u}_J + [K_{IJ}]u_J\} + \sum_{j=n_K+1}^{n_I} \frac{\phi_j \phi_j^T}{\omega_j^4} \{[M_{IJ}]D^2\ddot{u}_J + [K_{IJ}]\ddot{u}_J\} \quad (16)$$

It is convenient to write this expression in matrix form as

$$u_I = [\Phi_K \mid A_J \mid B_J \mid C_J] \begin{Bmatrix} \eta_K \\ u_J \\ \ddot{u}_J \\ D^2\ddot{u}_J \end{Bmatrix} \quad (17)$$

where $[\Phi_K]$ is the $(n_I \times n_K)$ matrix with the $n_K$ lower eigenvectors and

$$[A_J] = -[V_1]\,[K_{IJ}] \quad (18a)$$

$$[B_J] = [V_2]\,[K_{IJ}] - [V_1]\,[M_{IJ}] \quad (18b)$$

$$[C_J] = [V_2]\,[M_{IJ}] \quad (18c)$$

The following notation has been introduced in Eqs. (18a–18c):

$$[V_\ell] = [\Phi_R]\,[\Lambda_R]^{-\ell}\,[\Phi_R]^T, \qquad \ell = 1, 2, \ldots, \quad (19)$$

where $[\Phi_R]$ is the $(n_I \times n_R)$ matrix with the $n_R$ higher eigenvectors, and $[\Lambda_R]$ is the diagonal matrix with the corresponding $n_R$ higher eigenvalues $\lambda_j = \omega_j^2$.

The total displacement vector $u$ can be expressed as

$$u = \left[\begin{array}{c|c|c|c} \Phi_K & A_J & B_J & C_J \\ \hline 0 & I_J & 0 & 0 \end{array}\right] \begin{Bmatrix} \eta_K \\ u_J \\ \ddot{u}_J \\ D^2\ddot{u}_J \end{Bmatrix} \quad (20)$$

Equation (20) constitutes the basis of the proposed modal synthesis formulation. It allows the motion of the system to be expressed as a linear combination of a reduced number $(n_K + 3n_J)$ of generalized coordinates in terms of the lower vibration modes of the system calculated with fixed interface condition and other correction terms.

In the literature on substructure synthesis, it is customary to refer to all of these correction terms, other than the lower vibration modes $[\Phi_K]$, as the "component modes." These modes are introduced to account somehow for the effect of the truncated modes as well as the input from the adjacent substructure. Adopting the same terminology here also, we can designate the matrices $[A_J]$, $[B_J]$, and $[C_J]$ as the first-, second-, and third-order component mode matrices, respectively.

Equation (20) can now be used to establish a relationship between the proposed technique and the CBH method.[3,4] We first note that one can also define the matrices $[V_1]$ and $[V_2]$ in terms of the stiffness and mass matrices and the lower modes and frequencies of the substructure by using the identities

$$[K_{II}]^{-1} = [\Phi]\,[\Lambda]^{-1}\,[\Phi]^T \quad (21a)$$

$$[K_{II}]^{-1}\,[M_{II}]\,[K_{II}]^{-1} = [\Phi]\,[\Lambda]^{-2}\,[\Phi]^T \quad (21b)$$

With the identities in Eqs. (21a) and (21b), the matrices $[V_1]$ and $[V_2]$ appearing in Eq. (18) and defined by Eq. (19) can also be expressed as

$$[V_1] = [K_{II}]^{-1} - [\Phi_K]\,[\Lambda_K]^{-1}\,[\Phi_K]^T \quad (22a)$$

$$[V_2] = [K_{II}]^{-1}\,[M_{II}]\,[K_{II}]^{-1} - [\Phi_K][\Lambda_K]^{-2}\,[\Phi_K]^T \quad (22b)$$

Thus, matrix $[A_J]$ in Eq. (18a) that defines the contribution of the first-order component modes can be written as

$$[A_J] = -[K_{II}]^{-1}\,[K_{IJ}] + [\Phi_K][\Lambda_K]^{-1}\,[\Phi_K]^T\,[K_{IJ}] \quad (23)$$

In the CBH method, the total displacement $u$ in Eq. (20) is defined as

$$u = \left[\begin{array}{c|c} \Phi_K & -K_{II}^{-1}K_{IJ} \\ \hline 0 & I_J \end{array}\right] \begin{Bmatrix} \eta_K \\ u_J \end{Bmatrix} \quad (24)$$

Comparing Eqs. (20) and (24) and considering Eq. (23), the relationship between the proposed method and the CBH method becomes evident. First, the contribution of the higher modes ensuing from the matrices $[B_J]$ and $[C_J]$ are ignored in the latter method. Second, only one part of the first-order correction term associated with the first term in Eq. (23) is considered. The first-order correction term represents the pseudostatic response contribution of the truncated modes. In Eq. (23), this contribution is expressed as a difference of the pseudostatic contribution of all of the modes minus the pseudostatic contribution of the kept modes. In the CBH method, however, the part that represents the contribution of the kept modes, associated with the second term in Eq. (23) and expressed as $[\Phi_K]\,[\Lambda_K]^{-1}\,[\Phi_K]\,[K_{IJ}]u_J$, is absent. Thus, in their formulation, the pseudostatic response contribution of the truncated modes is replaced by the pseudostatic contribution of all of the modes; this latter contribution is associated with the first term in Eq. (23) and is given by the term $-K_{II}^{-1}K_{IJ}u_J$. However, it will be shown later in the Appendix that, despite the different transformation matrices used in the two formulations, the CBH method and the first-order correction version of the proposed method lead to the same results. Thus CBH method is the same as the first-order method in the approach proposed here.

The transformation in Eq. (20) can be used to reduce the order of the equations of motion (1). Substituting Eq. (20) into Eq. (1) and premultiplying the resulting equations of motion by the transpose of the transformation matrix, we obtain

$$[\mu]\ddot{q} + [\kappa]q = F \quad (25)$$

where

$$[\mu] = \begin{bmatrix} \mu_{11} & \mu_{12} & \mu_{13} & \mu_{14} \\ & \mu_{22} & \mu_{23} & \mu_{24} \\ \text{symm} & & \mu_{33} & \mu_{34} \\ & & & \mu_{44} \end{bmatrix}$$

$$[\kappa] = \begin{bmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} & \kappa_{14} \\ & \kappa_{22} & \kappa_{23} & \kappa_{24} \\ \text{symm} & & \kappa_{33} & \kappa_{34} \\ & & & \kappa_{44} \end{bmatrix} \quad (26)$$

$$q^T = [\eta_K^T;\ u_J^T;\ \ddot{u}_J^T;\ D^2u_J^T] \quad (27a)$$

$$F^T = [0_I^T;\ F_J^T;\ 0_J^T;\ 0_J^T] \quad (27b)$$

The submatrices $[\mu_{ij}]$ are

$$[\mu_{11}] = [\Phi_K^T]\,[M_{II}]\,[\Phi_K] \quad (28a)$$

$$[\mu_{12}] = [\Phi_K^T]\,[M_{II}A_J + M_{IJ}] \quad (28b)$$

$$[\mu_{13}] = [\Phi_K^T M_{II} B_J] \quad (28c)$$

$$[\mu_{14}] = [\Phi_K^T M_{II} C_J] \quad (28d)$$

$$[\mu_{22}] = [A_J^T M_{II} A_J] + [A_J^T M_{IJ}] + [A_J^T M_{IJ}]^T + [M_{JJ}] \quad (28e)$$

$$[\mu_{23}] = [A_J^T M_{II} + M_{JI}]\,[B_J] \quad (28f)$$

$$[\mu_{24}] = [A_J^T M_{II} + M_{JI}]\ [C_J] \tag{28g}$$

$$[\mu_{33}] = [B_J^T M_{II} B_J] \tag{28h}$$

$$[\mu_{34}] = [B_J^T M_{II} C_J] \tag{28i}$$

$$[\mu_{44}] = [C_J^T M_{II} C_J] \tag{28j}$$

The submatrices $\kappa_{ij}$ are similarly defined by replacing the matrices $[M_{II}]$, $[M_{IJ}]$, and $[M_{JJ}]$ by $[K_{II}]$, $[K_{IJ}]$, and $[K_{JJ}]$, respectively. Using the orthonormal properties of the eigenvectors $[\Phi_K]$ and $[\Phi_R]$ and the definitions of $[A_J]$, $[B_J]$, and $[C_J]$ in Eqs. (18a-18c), the submatrices $[\mu_{ij}]$ can be further simplified as

$$[\mu_{11}] = [I_K] \tag{29a}$$

$$[\mu_{12}] = [\Phi_K^T M_{IJ}] \tag{29b}$$

$$[\mu_{13}] = [\mu_{14}] = 0 \tag{29c}$$

$$[\mu_{22}] = [M_{JJ} + K_{IJ}^T V_2 K_{IJ} - K_{IJ}^T V_1 M_{IJ} - (K_{IJ}^T V_1 M_{IJ})^T] \tag{29d}$$

$$[\mu_{23}] = [-K_{IJ}^T V_3 K_{IJ} - M_{IJ}^T V_1 M_{IJ} + K_{IJ}^T V_2 M_{IJ} + (K_{IJ}^T V_2 M_{IJ})^T] \tag{29e}$$

$$[\mu_{24}] = [M_{IJ}^T V_2 - K_{IJ}^T V_3]\ [M_{IJ}] \tag{29f}$$

$$[\mu_{33}] = [K_{IJ}^T V_4 K_{IJ} + M_{IJ}^T V_2 M_{IJ} - K_{IJ}^T V_3 M_{IJ} - (K_{IJ}^T V_3 M_{IJ})^T] \tag{29g}$$

$$[\mu_{34}] = [K_{IJ}^T V_4 - M_{IJ}^T V_3]\ [M_{IJ}] \tag{29h}$$

$$[\mu_{44}] = [M_{IJ}^T V_4 M_{IJ}] \tag{29i}$$

Proceeding in a similar fashion, it can be shown that the submatrices $[\kappa_{ij}]$ can be written as

$$[\kappa_{11}] = [\Lambda_K] \tag{30a}$$

$$[\kappa_{12}] = [\Phi_K^T K_{IJ}] \tag{30b}$$

$$[\kappa_{13}] = [\kappa_{14}] = 0 \tag{30c}$$

$$[\kappa_{22}] = [K_{JJ}] - [K_{IJ}^T V_1 K_{IJ}] \tag{30d}$$

$$[\kappa_{23}] = [\kappa_{24}] = 0 \tag{30e}$$

$$[\kappa_{33}] = [K_{IJ}^T V_3 K_{IJ} + M_{IJ}^T V_1 M_{IJ} - K_{IJ}^T V_2 M_{IJ} - (K_{IJ}^T V_2 M_{IJ})^T] \tag{30f}$$

$$[\kappa_{34}] = [K_{IJ}^T V_3 - M_{IJ}^T V_2]\ [M_{IJ}] \tag{30g}$$

$$[\kappa_{44}] = [M_{IJ}^T V_3 M_{IJ}] \tag{30h}$$

## Substructures' Coupling

The next step in the development of the modal synthesis method is the coupling of the equations of motion of each substructure written as Eq. (25). To simplify the presentation of the coupling process, we will consider that the complete structure has been divided into two substructures. The substructures will be designated as substructure a and substructure b. The superscripts and subscripts $a$ and $b$ will denote a quantity associated with the corresponding substructure.

The equations of motion (25) for the two substructures can be combined to write the equations of motion for the system in uncoupled form as

$$\left[\begin{array}{c|c} \mu_a & 0 \\ \hline 0 & \mu_b \end{array}\right]\begin{Bmatrix} \ddot{q}_a \\ \ddot{q}_b \end{Bmatrix} + \left[\begin{array}{c|c} \kappa_a & 0 \\ \hline 0 & \kappa_b \end{array}\right]\begin{Bmatrix} q_a \\ q_b \end{Bmatrix} = \begin{Bmatrix} F_a \\ F_b \end{Bmatrix} \tag{31}$$

The vectors and matrices in Eq. (31) are defined in Eqs. (25-30).

The equations of motion can be coupled by imposing the kinematic and kinetic compatibility conditions:

$$u_J^a = u_J^b \tag{32}$$

$$F_J^a = -F_J^b \tag{33}$$

For this we introduce the following coordinate transformation in Eq. (31):

$$\begin{Bmatrix} q_a \\ q_b \end{Bmatrix} = \left[\begin{array}{ccccc} I_K^a & 0 & 0 & 0 & 0 \\ 0 & 0 & I_J & 0 & 0 \\ 0 & 0 & 0 & I_J & 0 \\ 0 & 0 & 0 & 0 & I_J \\ \hline 0 & I_K^b & 0 & 0 & 0 \\ 0 & 0 & I_J & 0 & 0 \\ 0 & 0 & 0 & I_J & 0 \\ 0 & 0 & 0 & 0 & I_J \end{array}\right]\begin{Bmatrix} \eta_\kappa^a \\ \eta_\kappa^b \\ u_J \\ \ddot{u}_J \\ D^2\ddot{u}_J \end{Bmatrix} = \begin{bmatrix} T_a \\ T_b \end{bmatrix} x \tag{34}$$

Premultiplying the resulting equation by $[T_a^T \mid T_b^T]$, we obtain

$$[\tilde{M}]\ddot{x} + [\tilde{K}]x = g(t) \tag{35}$$

where

$$[\tilde{M}] = [T_a]^T [\mu_a] [T_a] + [T_b]^T [\mu_b] [T_b] \tag{36a}$$

$$[\tilde{K}] = [T_a]^T [\kappa_a] [T_a] + [T_b]^T [\kappa_b] [T_b] \tag{36b}$$

$$g(t) = [T_a]^T F_a + [T_b]^T F_b \tag{36c}$$

Carrying out the matrix multiplications indicated in Eqs. (36a) and (36b), the reduced matrices $[\tilde{M}]$ and $[\tilde{K}]$ can be defined as

$$[\tilde{M}] = \begin{bmatrix} I_k^a & 0 & \Phi_k^{aT} M_{IJ}^a & 0 & 0 \\ & I_k^b & \Phi_k^{bT} M_{IJ}^b & 0 & 0 \\ & & \tilde{m}_{33} & \tilde{m}_{34} & \tilde{m}_{35} \\ \text{symm} & & & \tilde{m}_{44} & \tilde{m}_{45} \\ & & & & \tilde{m}_{55} \end{bmatrix} \tag{37a}$$

$$[\tilde{K}] = \begin{bmatrix} \Lambda_k^a & 0 & \Phi_k^{aT} K_{IJ}^a & 0 & 0 \\ & \Lambda_k^b & \Phi_k^{bT} K_{IJ}^b & 0 & 0 \\ & & \tilde{k}_{33} & 0 & 0 \\ \text{symm} & & & \tilde{k}_{44} & \tilde{k}_{45} \\ & & & & \tilde{k}_{55} \end{bmatrix} \tag{37b}$$

where

$$[\tilde{m}_{33}] = \sum_{r=a,b} [M_{JJ}^r + K_{IJ}^{rT} V_2^r K_{IJ}^r - K_{IJ}^{rT} V_1^r M_{IJ}^r - (K_{IJ}^{rT} V_1^r M_{IJ}^r)^T] \tag{38a}$$

$$[\tilde{m}_{34}] = -\sum_{r=a,b} [K_{IJ}^{rT} V_3^r K_{IJ}^r + M_{IJ}^{rT} V_1^r M_{IJ}^r - K_{IJ}^{rT} V_2^r M_{IJ}^r - (K_{IJ}^{rT} V_2^r M_{IJ}^r)^T] \tag{38b}$$

$$[\tilde{m}_{35}] = \sum_{r=a,b} [M_{IJ}^{rT} V_2^r - K_{IJ}^{rT} V_3^r]\ [M_{IJ}^r] \tag{38c}$$

$$[\tilde{m}_{44}] = \sum_{r=a,b} [K_{IJ}^{rT} V_4^r K_{IJ}^r + M_{IJ}^{rT} V_2^r M_{IJ}^r - K_{IJ}^{rT} V_3^r M_{IJ}^r - (K_{IJ}^{rT} V_3^r M_{IJ}^r)^T] \tag{38d}$$

$$[\tilde{m}_{45}] = \sum_{r=a,b} [K_{IJ}^{rT} V_4^r - M_{IJ}^{rT} V_3^r]\ [M_{IJ}^r] \tag{38e}$$

$$[\tilde{m}_{55}] = \sum_{r=a,b} [M_{IJ}^{rT} V_4^r M_{IJ}^r] \tag{38f}$$

$$[\tilde{k}_{33}] = \sum_{r=a,b} [K_{JJ}^r - K_{IJ}^{rT} V_1^r K_{IJ}^r] \tag{38g}$$

$$[\tilde{k}_{44}] = \sum_{r=a,b} [K_{IJ}^{rT} V_3^r K_{IJ}^r + M_{IJ}^{rT} V_1 M_{IJ}^r - K_{IJ}^{rT} V_2^r M_{IJ}^r - (K_{IJ}^{rT} V_2^r M_{IJ}^r)^T] \tag{38h}$$

$$[\tilde{k}_{45}] = \sum_{r=a,b} [K_{IJ}^{rT} V_3^r - M_{IJ}^{rT} V_2^r] [M_{IJ}^r] \tag{38i}$$

$$[\tilde{k}_{55}] = \sum_{r=a,b} [M_{IJ}^{rT} V_3^r M_{IJ}^r] \tag{38j}$$

It is straightforward to show that the forcing function term $g(t)$ defined in Eq. (36c) vanishes. Therefore, Eq. (35) corresponds to the reduced-order free vibration equations of motion of the combined structural system.

The natural frequencies of the system are obtained from the solution of the associated eigenvalue problem of order $(n_k^a + n_k^b + 3n_J)$:

$$[\tilde{M}]\tilde{\psi}_j = \lambda_j [\tilde{K}]\tilde{\psi}_j, \qquad j = 1, \ldots, n_k^a + n_k^b + 3n_J \tag{39}$$

where $\tilde{\psi}_j$ are the eigenvectors of the transformed condensed system. The eigenvectors of the original structure can be calculated applying the transformations of Eqs. (20) and (34) in reverse order. That is,

$$\psi_j = \begin{Bmatrix} \psi_I^a \\ \hline \psi_J \\ \hline \psi_I^b \end{Bmatrix}_j = \left[\begin{array}{c|c|c|c|c} \Phi_k^a & 0 & A_J^a & B_J^a & C_J^a \\ \hline 0 & 0 & I_J & 0 & \\ \hline 0 & \Phi_K^b & A_J^b & B_J^b & C_J^b \end{array}\right] \tilde{\psi}_j \tag{40}$$

In Eq. (40), the modal vectors of the combined system $\psi_j$ have been divided into three parts to help interpret the previous transformation. The vector $\psi_j$ contains the elements of $\psi$ associated with the $n_J$ interface coordinates, and therefore it is common to both substructures. The vectors $\psi_I^a$ and $\psi_I^b$ correspond to the modal coordinates associated with the interior degrees of freedom of the substructures a and b.

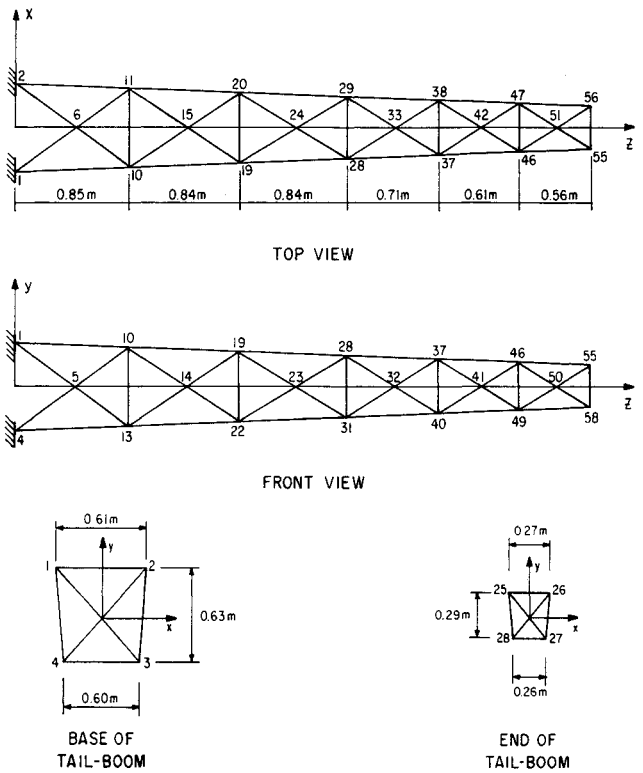


**Fig. 2 Details of the structure considered in the example problem.**

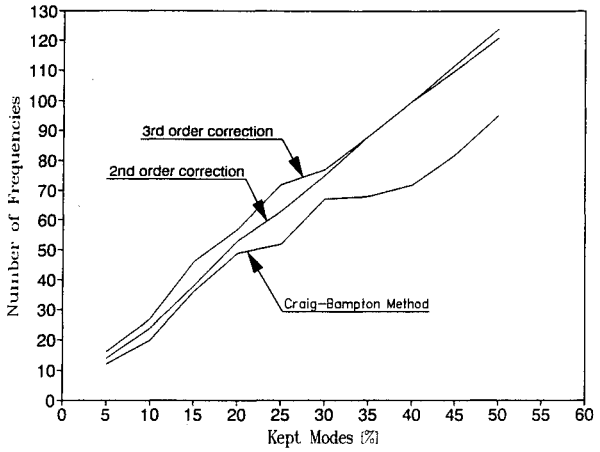


**Fig. 3 Number of frequencies with error less than 1% vs the percent of substructure modes used in synthesis.**

Here it is relevant to note that the size of the final eigenvalue problem is $n_k^a + n_k^b + 3n_j$ if all terms up to third-order correction terms are included. This size, however, can be reduced by excluding the higher component modes (i.e., the higher order terms), but such a reduction will be associated with a loss in the accuracy of the calculated results. An improvement in the accuracy of the calculated eigenproperties can also be achieved by including a larger number of the normal modes of each substructure, that is, by increasing $n_k^a$ and $n_k^b$. Whether or not the scheme of using more normal modes and less component modes will be more efficient than the scheme of using less normal modes and more component modes will depend on the size of the substructures (which determines the amount of effort required to calculate their additional normal modes) and also on the number of degrees of freedom at the interface.

## Numerical Results

The structure selected to present numerical results of the proposed formulation is shown in Fig. 2. The model of the structure corresponds to the open tail-boom structure of the U.S. Army helicopter analyzed by Arora and Nguyen.[18] The structure is idealized here as a three-dimensional frame with 168 elements and a total of 324 degrees of freedom. The members of the structure are made of aluminum tubular sections with a cross-sectional area of 1.27 in.$^2$, transverse and polar moments of inertia of 0.668 in.$^4$ and 1.1164 in.$^4$, respectively, and Young's modulus of $10.5 \times 10^6$ psi. The structure is divided into two substructures as indicated in Fig. 1. The number of degrees of freedom of the substructures a and b with the interface coordinates fixed are 138 and 162, respectively. The number of degrees of freedom at the substructures' interface is 24.

Figure 3 shows the number of natural frequencies calculated with a relative error less than 1% vs the percentage of vibration modes used to represent each substructure. In all of the examples, the same percentage of kept modes has been used for both substructures. The relative error in the frequencies is defined as

$$\epsilon_j = \frac{|\omega_{ex} - \omega_{app}|}{\omega_{ex}} \times 100; \qquad j = 1, 2, \ldots$$

where $\omega_{ex}$ is the $j$th frequency obtained by solving the full eigenproblem and $\omega_{app}$ is the value obtained using a modal synthesis formulation. It is observed that the second- and third-order correction methods do improve the accuracy of the frequencies calculated by the first-order or CBH method, although these higher order terms are associated with a larger final eigenvalue problem.

Figure 4 shows the number of modal shape vectors calculated with an error less than 1% as a function of the percent-

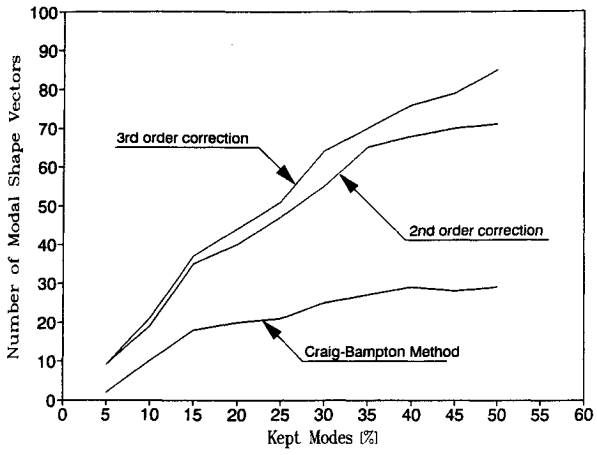


**Fig. 4 Number of eigenvectors with rms error less than 1% vs the percent of substructure modes used in synthesis.**

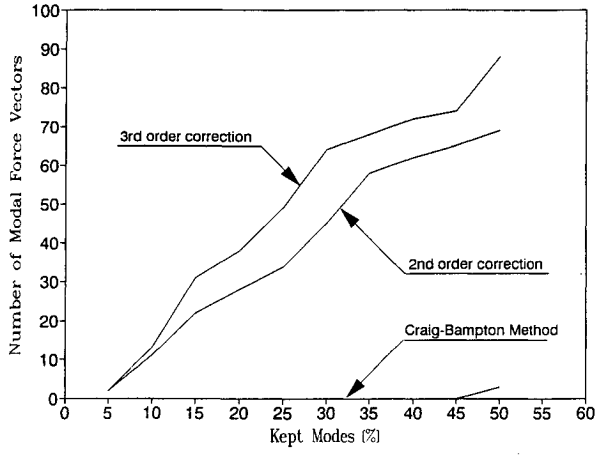


**Fig. 5 Number of modal forces with error less than 1% vs the percent of substructure modes used in synthesis.**

age of the kept modes for both substructures. Two definitions were used to calculate the relative error for a vector $X$:

Absolute value error:

$$\epsilon = \frac{|\sum_i |(X_{ex})_i - (X_{app})_i||}{|\Sigma |(X_{ex})_i||} \times 100$$

Root-mean-square error:

$$\epsilon = \frac{\{\sum_i [(X_{ex})_i - (X_{app})_i]^2\}^{1/2}}{\{\sum_i (X_{ex})_i^2\}^{1/2}} \times 100$$

where $(X_{ex})_i$ is the $i$th element of the vector $X_{ex}$ obtained from the solution of the full eigenproblem and $(X_{app})_i$ is the corresponding value obtained using a modal synthesis method. The errors calculated with both definitions were almost the same, and therefore the root-mean-square error is used to present the numerical results. The proposed formulation with the second- and third-order correction terms is seen to predict results with very good accuracy. However, comparing Figs. 3 and 4, one would note that the modes are not calculated with the same accuracy as the frequencies.

The accuracy of the results obtained with the proposed formulation is even more pronounced when the modal force vectors are compared. The $j$th modal force vector of the complete structure is defined as the product of the global stiffness matrix and the $j$th eigenvector. Figure 5 shows the number of modal force vectors with error less than 1% calculated with the three methods vs the percentage of the kept modes. Comparing this with Figs. 3 and 4, one notes that the accuracy of the modal force vectors is significantly smaller than those of the frequencies and eigenvectors. In fact, in this example problem the first-order formulation or CBH approach is seen to provide rather poor results. A large number of substructures' modes were needed to begin obtaining accurate results in the first-order or CBH approach. However, the accuracy of the results is significantly improved when the second- and third-order terms are included in the mode synthesis. It should be mentioned that the CBH or first-order correction method does converge to the exact modal force vector as the number of kept modes are increased. That is, when all of the modes of the substructures were used, the exact solution was obtained.

## Concluding Remarks

An improved mode synthesis approach is proposed to reduce the error in the eigenproperties of a structural system caused by the use of a limited number of fixed interface normal modes of substructures. The improvement is achieved by utilization of a higher order mode superposition method that accounts for the truncated modes without explicitly calculating them. Depending on the order of the correction terms used, one obtains a first-, second-, or third-order mode synthesis approach. It is shown that the popular Craig-Bampton-Hurty approach is a first-order approach wherein only the first-order correction terms are retained to account for the truncated modes. It is shown that the retention of the higher order correction terms, or equivalently the inclusion of more component modes, does improve the accuracy of the calculated modal properties, although it also increases the size of the final eigenvalue problem. A similar improvement in the accuracy of the calculated modal properties can also be realized by including a larger number of normal modes from each substructure with a lower order formulation. Thus, obviously there is a tradeoff involved here. The answer to the question as to which of the two approaches—the one with more normal modes but less component modes or the other with less normal modes but more component modes—is better will, of course, depend on the effort involved in calculating the additional normal modes of the substructures and the number of the degrees of freedom at the interface. The method proposed in this paper improves the accuracy without calculating any additional normal modes but by including more component modes that do not require any knowledge of the higher normal modes. But this improvement does not come by without any additional effort as it increases the size of the final eigenvalue problem. However, it is still expected that, for the same level of accuracy in the calculated modal properties, a higher order approach will compare favorably with the first-order approach (CBH method) for large substructures connected by a relatively small number of degrees of freedom at their interface.

## Appendix

### Equivalence Between the First-Order Formulation and the CBH Method

If only the first correction term is used in Eq. (20) to represent the motion of the substructure, the reduced equations of motion (25) assume the form

$$\begin{bmatrix} I_K & \Phi_K^T M_{IJ} \\ M_{IJ}^T \Phi_K & \mu_{22} \end{bmatrix} \ddot{q} + \begin{bmatrix} \Lambda_K & \Phi_K^T K_{IJ} \\ K_{IJ}^T \Phi_K & \kappa_{22} \end{bmatrix} q = F \qquad \text{(A1)}$$

The equivalent equation of motion of reduced order for the CBH method can be obtained substituting the transformation (24) in Eq. (1) and premultiplying by the transverse of the same transformation matrix. It can be shown that this leads to

$$\begin{bmatrix} I_K & \Phi_K^T M_{IJ} - H_{KJ} \\ M_{IJ}^T \Phi_K - H_{KJ}^T & \mu_{22}^* \end{bmatrix} \ddot{q}^* + \begin{bmatrix} \Lambda_k & 0 \\ 0 & \kappa_{22}^* \end{bmatrix} q^* = F \qquad \text{(A2)}$$

where

$$H_{KJ} = \Lambda_K^{-1} \Phi_K^T K_{IJ} \qquad \text{(A3)}$$

$$\mu_{22}^{*} = M_{JJ} + K_{IJ}^{T}K_{II}^{-1}M_{II}K_{II}^{-1}K_{IJ} - K_{IJ}^{T}K_{II}^{-1}M_{IJ} - M_{IJ}^{T}K_{II}^{-1}K_{IJ} \quad \text{(A4)}$$

$$\kappa_{22}^{*} = K_{JJ} - K_{IJ}^{T}K_{II}^{-1}K_{IJ} \quad \text{(A5)}$$

The two equations of motion (A1) and (A2) are related through similarity transformations. The vector $q^*$ in Eq. (A2) can be expressed in terms of the vector $q$ in Eq. (A1):

$$q^{*} = \begin{bmatrix} I_k & H_{KJ} \\ 0 & I_J \end{bmatrix} q \quad \text{(A6)}$$

Substituting Eq. (A6) into Eq. (A2) and premultiplying by the transpose of the transformation matrix, one obtains

$$\begin{bmatrix} I_K & \Phi_K^{T}M_{IJ} \\ M_{IJ}^{T}\Phi_K & \hat{\mu}_{22} \end{bmatrix} \ddot{q} + \begin{bmatrix} \Lambda_K & \Lambda_K H_{KJ} \\ H_{KJ}^{T}\Lambda_K & \hat{\kappa}_{22} \end{bmatrix} q = F \quad \text{(A7)}$$

where

$$\hat{\mu}_{22} = \mu_{22}^{*} - H_{KJ}^{T}H_{KJ} + H_{KJ}^{T}\Phi_K^{T}M_{IJ} + M_{IJ}^{T}\Phi_K H_{KJ} \quad \text{(A8)}$$

$$\hat{\kappa}_{22} = \kappa_{22}^{*} + H_{KJ}^{T}\Lambda_K H_{KJ} \quad \text{(A9)}$$

Using the definition of matrix $H_{KJ}$ in Eq. (A3), carrying out the matrix multiplication indicated in Eqs. (A8) and (A9), and recalling the definitions of matrices $[V_1]$ and $[V_2]$ in Eqs. (22a and 22b), it is straightforward to show that

$$\Lambda_k H_{KJ} = \Phi_k^{T}K_{IJ} \quad \text{(A10)}$$

$$\hat{\mu}_{22} = M_{JJ} + K_{IJ}^{T}V_2K_{IJ} - K_{IJ}^{T}V_1M_{IJ} - M_{IJ}^{T}V_1K_{IJ} = \mu_{22} \quad \text{(A11)}$$

$$\hat{\kappa}_{22} = K_{IJ} + K_{IJ}^{T}V_1K_{IJ} = \kappa_{22} \quad \text{(A12)}$$

Therefore, Eqs. (A1) and (A2) are indeed related by the transformation (A6). The equations of motion (A1) or (A2) for the two substructures have to be combined to obtain the free vibration equations of motion of the combined system. Since the same coupling procedure is applicable to both methods, this will not alter the equivalence between the two formulations demonstrated earlier. Finally, since the mass and stiffness matrices in the reduced-order eigenproblem (39) of both methods are related through similarity transformations, the two methods will lead to the same set of eigenvalues and eigenvectors for the combined structure.

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